交通流モデルとしての双安定性をもつ非線形差分方程式

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概要

交通流を記述する数理モデルの中でも，巨視的モデルは，交通ネットワークシミュレーションなどの大規模な数値計算において，計算コストを低減できるという利点がある．本研究では，交通流に実験的に見られる一般的な特徴である「双安定性」に着目し，交通流を巨視的に記述する新しい非線形差分方程式を提案した．その結果，一様解の安定性という観点から，本モデルのパラメータが交通渋滞を解消する可能性があることがわかった．さらに、提案モデルと他の交通流モデルとの対応関係を調べるために超離散極限と連続極限の結果を示した。

Nonlinear difference equation with bi-stability as a new traffic flow model

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Abstract

Among mathematical traffic flow models, macroscopic models have the advantage of reducing computational costs in large-scale numerical calculations such as transportation network simulations. In this study, we focus on bi-stability, which is a general feature empirically observed in a traffic flow, and propose a new nonlinear difference equation that describes a traffic flow at the macroscopic level. We found that the parameters in our model have the potential to eliminate traffic jams from the perspective of stability of a homogeneous flow. Moreover, ultra-discrete and continuous limit results are presented to investigate the correspondence between the proposed model and other models.

1 Introduction

The dynamics of traffic flow includes instabilities as the average traffic density increases. In other words, a free flow is stable when the vehicle density is low, but becomes unstable when the vehicle density exceeds the critical density. In the latter case, the traffic flow transits from a free flow into another stable state, which corresponds to a jamming flow observed as a spontaneous traffic jam. As an example, Gasser et al. [1] proved that the instability of a free flow is understood by Hopf bifurcation; they numerically showed the global bifurcation structure possessed by the optimal velocity (OV) model [2]. The OV model is a car-following model describing an adaptation to the optimal velocity that depends on the headway between two neighboring vehicles; it is well known for its successful explanation of the mechanism of spontaneous traffic jams. The results of Gasser et al. show that the Hopf bifurcation behavior is locally supercritical, but macroscopically exhibits a subcritical structure. The optimal velocity model reproduces "bi-stability" at a certain density region, in which free and jamming flows coexist. It should be emphasized here that this bi-stability is a general feature empirically observed in traffic flows [3]. Many mathematical models have been proposed to reproduce traffic flow. These models can be approximately classified into three types: (i) density wave model/gas-kinetic models (partial differential equation), (ii) car-following models (ordinary differential equation), and (iii) cellular automaton models (max-plus algebra/master equation/rule-based algorithm). Our interest is in designing a precise traffic flow model and the relationship between these descriptions. For example, the cellular automaton model derived from partial differential equation with bi-stability. Furthermore, the correspondence also facilitate the development of a mathematical analysis method for a cellular automaton model for which analysis methods have not been developed. Clarifying the correspondence, and understanding the structure of the solution, are extremely significant from an algebraic point of view. Hence, the aim of this study is to propose and analyze a new nonlinear difference equation with bi-stability, and to then show both models obtained by continuous and ultradiscrete limits.

2 A new mathematical model considering past weights

2.1 Proposed model

We propose the following nonlinear difference equation as a new model:

$$\rho_x^{t+\Delta t} = \rho_x^t - \rho_x^t b_x^t + \rho_{x-\Delta x}^t b_{x-\Delta x}^t \tag{1a}$$

$$b_x^t = (1-\rho_{x+\Delta x}^t) \times \left\{1 - \left((1-\alpha)\rho_x^{t-\Delta t} + \alpha\rho_{x+\Delta x}^{t-\Delta t}\right)\right\}, \tag{1b}$$

where ρ_x^t is the density at a lattice x and a time t and α is the parameter takes the value $(0,1)$ and b_x^t is the transition rate of vehicles at a lattice x and a time t. In this model, one-dimensional road is divided into a lattice, and the conservation law of vehicles for each lattice is described. The effect of $\left\{1 - \left((1-\alpha)\rho_x^{t-\Delta t} + \alpha\rho_{x+\Delta x}^{t-\Delta t}\right)\right\}$ is to decide whether to move to the lattice in front depending on the congestion in the vicinity. The effect of $(1-\rho_{x+\Delta x}^t)$ is to brake suddenly when there is a leading can $\rho_{x+\Delta x}^t = 1$. α represents the strength of the dependence on congestion in the vicinity of the driver's own location.

2.2 Numerical experiments and linear stability analysis for the new model

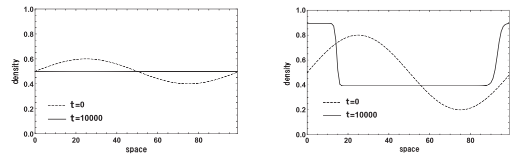
We focus on the bi-stability as the model to confirm the validity of the traffic flow model. First we numerically investigate a homogeneous flow with a perturbation. In our numerical experiments, we impose the periodic boundary condition, and then set $\Delta x = 1, \Delta t = 1$, $x \in \{1,2,3,\cdots,L\}$, where L is the number of lattices. The initial value of ρ_x^0 is set as follows:

$$\rho_x^0 = \rho_0 + A\sin\frac{2\pi x}{L}, \tag{2}$$

where ρ_0 is the density at $x \in \{1,2,3,\cdots,L\}$, and A is the amplitude of the perturbation. We assume that $\rho_x^0 = \rho_x^1$ as the initial value.

We found that the model (1) shows bi-stability at least at $\rho_0 = 0.5$. It was found that, if the perturbation is small, the density wave converges to the homogeneous solution; However, if the perturbation is not small, the density wave converges to a traveling wave solution, which is expected to be another solution. In the latter case, as shown in Fig.1 (b), the initial perturbation grows into a traveling wave. This traveling wave solution is observed to propagate in the direction opposite to the direction in which the cars move. This feature is consistent with the fact that a jam cluster in a real traffic jam propagates backward [4].

Next, a linear stability analysis was performed on the model. The bi-stable region was investi-



(a) $(\rho_0, A) = (0.5, 0.1)$ (b) $(\rho_0, A) = (0.5, 0.3)$

Figure1: Numerical solutions fortwo cases (a) $A = 0.1$, (b) $A = 0.3$ with $\rho_0 = 0.5, \alpha = 0.2$. The dashed line is the initial value, and the solid line is the converging numerical solution plotted at $t = 10,000$.

gated numerically. The results are shown in Fig. 2. From these analyses, it was found that the unstable region of the homogeneous solution disappears for $\alpha > 0.401$. This suggests that by controlling the parameter α in the real traffic flow, it is possible to always maintain the free flow.

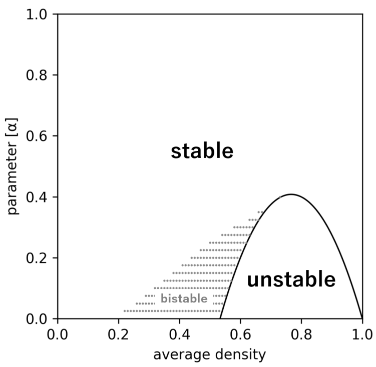


Figure2: Graph with $L = 100$. The solid curve represents the neutral stability obtained by the linear stability analysis. The points represent the bi-stable region, which is obtained numerically.

3 Ultra-discrete and continuous limit

3.1 Ultra-discrete limit

As one of our aims, we would like to see the correspondence between the ultra-discretized model and the cellular automaton model. Using the variable transformations $\rho_x^t = e^{-U_j^n/\varepsilon}, 1-\rho_x^t = e^{-V_j^n/\varepsilon}, \alpha = e^{-A/\varepsilon}, 1-\alpha = e^{-B/\varepsilon}$, and taking the limit $\varepsilon \to +0$, we obtain the following ultra-discrete equations:

$$\begin{aligned} U_j^{n+1} = \min\Big[& U_j^n + U_{j+1}^n, B + V_{j+1}^n + U_j^n + U_j^{n-1}, \\ & A + V_{j+1}^n + U_j^n + U_{j+1}^{n-1}, \\ & A + U_{j-1}^n + V_j^n + V_j^{n-1}, \\ & B + U_{j-1}^n + V_j^n + V_{j-1}^{n-1}\Big], \end{aligned} \tag{3a}$$

$$\begin{aligned} V_j^{n+1} = \min\Big[& V_{j-1}^n + V_j^n, A + V_{j+1}^n + V_{j+1}^{n-1} + U_j^n, \\ & B + V_{j+1}^n + V_j^{n-1} + U_j^n, \\ & B + V_j^n + U_{j-1}^n + U_{j-1}^{n-1}, \\ & A + V_j^n + U_{j-1}^n + U_j^{n-1}\Big], \end{aligned} \tag{3b}$$

$$0 = \min\Big[U_j^n, V_j^n\Big], \tag{3c}$$

$$0 = \min\Big[A, B\Big], \tag{3d}$$

where $\{0, \infty\}$ in U_j^n and V_j^n correspond to $\{1, 0\}$ in the cellular automaton models. Note that $U_j^n, V_j^n, A, B \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ owing to the inequality $0 \leq \rho_x^t, \alpha \leq 1$.

3.2 Continuous limit

We are also interested in the correspondence between our model (1) and previous density wave models.

We perform the Taylor expansion for (1a) and take the terms of the second order. If we take the limit $\Delta x \to 0$, $\Delta t \to 0$, keeping the maximum velocity $V_{\max} = \Delta x/\Delta t$ finite, we obtain

$$\frac{\partial \rho}{\partial t} = -V_{\max}\frac{\partial(\rho b)}{\partial x} \tag{4}$$

where $b = b(x,t)$ is the limit of b_x^t. Setting $v(x,t) = V_{\max} b(x,t)$, we obtain the continuous equation:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v)}{\partial x}. \tag{5}$$

Substitution (1a) into (1b) gives

$$\begin{aligned} b_x^{t+\Delta t} = & \Big\{1 - (\rho_x^t - \rho_x^t b_x^t + \rho_{x-\Delta x}^t b_{x-\Delta x}^t)\Big\} \\ & \times\Big\{1 - \Big((1-\alpha)\rho_x^t + \alpha\rho_{x+\Delta x}^t\Big)\Big\}. \end{aligned} \tag{6}$$

Adding $-b_x^t + (b_{x+\Delta x}^t - b_{x-\Delta x}^t)/2$ to both sides of (6), performing a Taylor expansion, and setting

$1/\tau \stackrel{\text{def}}{=} \Delta x/\Delta t^2$, we obtain

$$\begin{aligned}\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=&V_{\max}(1-\rho)\\ &\times(v-2V_{\max}\rho^2-3V_{\max}-\alpha V_{\max})\frac{\partial \rho}{\partial x}\\ &+V_{\max}\rho(1-\rho)\frac{\partial v}{\partial x}+\frac{1}{V_{\max}}\frac{V(\rho)-v}{\tau},\end{aligned} \tag{7}$$

where $V(\rho)=V_{\max}(1-\rho)^2$, which is the optimal velocity in a homogeneous flow. Thus, the continuous limit of model (1) is

$$\frac{\partial \rho}{\partial t}=-\frac{\partial(\rho v)}{\partial x}, \tag{8a}$$

$$\begin{aligned}\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=&V_{\max}(1-\rho)\\ &\times(v-2V_{\max}\rho^2-3V_{\max}-\alpha V_{\max})\frac{\partial \rho}{\partial x}\\ &+V_{\max}\rho(1-\rho)\frac{\partial v}{\partial x}+\frac{1}{V_{\max}}\frac{V(\rho)-v}{\tau}.\end{aligned} \tag{8b}$$

Note that the third terms on the right-hand side correspond to the self-driven force (inner-force) term used in the models of previous studies [5, 6].

4 Conclusion

In this study, we proposed a new traffic flow model described by a nonlinear difference equation and investigated the model both numerically and analytically. As a result, we analytically obtained the linear stability condition and numerically clarified the existence of the bi-stability region, where the solutions converge to different flows depending on the magnitude of the initial perturbation. Moreover, the parameters in our model have the potential to eliminate traffic jams from the perspective of a homogeneous flow stability. We also investigated the correspondences of our model with other models by taking two limits, that is, the ultradiscrete limit and a continuous limit. However, the specific correspondence with other CA models and density wave models has yet to be confirmed. Future studies will include analytically finding the bi-stability region and confirming a more detailed correspondence between the ultradiscrete and continuous limits.

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