自己駆動多体系の複スケール解析

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概要

生物集団の一般モデルとして Swarm Oscillators という数理モデルが田中ダン氏により提案された。これはスパイラル状及びターゲット状の凝集過程を示すことから、キイロタマホコリカビのような化学物質のスパイラル波を出しながら集まる過程を表現できると期待されると考えた。本論文ではこの現象を解析するための計算手法を提案する。凝集過程の初期段階のシナリオを説明するとともに、対象数理モデルで現れる波の波長、スパイラルの場合は回転角速度のモデルを決めるパラメータに対する依存性を明らかにした。

Multi time scale dynamics of a self propelled many body system

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Abstract

A General representation of a collective motion "Swarm Oscillators" was derived by D.Tanaka in 2007. The model shows aggregation movement accompanied by a spiral and target pattern wave, which is important in the case of real organisms such as *Dictyostelium*.To understand this behavior, we proposed how to analyse the spiral and target aggregation movement in this paper in a manner similar to that of Kuramoto model. It's shown that the senario of the first term of the aggregation process can be expressed by this model. And parameter dependence of a wavelength and rotation speed in spiral wave case, and that of a wavelength in target wave case is also clearified in this paper.

1 Introduction

Understanding of a collective motion of organisms with mathematical model is one of the most important topics in physics and information science. Various type of population can be our target such as fish, birds, slime molds and so on. [1]

To discuss such a collective motion in a general and mathematically reasonable way, D.Tanaka derived a normal form which represents a wide variety of collective motions, that is, "Swarm Oscillators".[2] Up to the choice of the parameters, we can make various types of behaviors in silico with this model. For instance, a motion like a flowing liquid, an ordered cristal, an aggregation movement with target and spiral pattern phase wave, what's more, a motion like a flexible membrane and an irregular motion can also be made. Considering the strategy of population , it seems notably important to focus on aggregation process. *D.discoideum*, which is one of cellular slime molds, grows as being separate under plenty of foods. In a bad condition such as starvation, they take a efficient strategy of migration. Forming a multicellular mold, changing to slug like shape, and moving to another place like as an individual.[3] Although this model seems fruitful, investigation of the model is far from satisfactory. In this paper, we proposed how to construct the solution of the aggregation movement accompanied by spiral and target pattern phase wave in this model. We got a quantitative and qualitative result of a relation between "wave length and wave propagation speed" and "molds' transfer movement".

2 Methods & Results

2.1 Summary of Swarm Oscillator

The Swarm Oscillator model was derived from a general "Chemotactic Oscillators"; a general model of population of cells in which each cells can communicate each other using a signal of some chemical substance. It was derived by a manipulation of phase reduction under a hypothesis of Hopf bifurcation. His result was given as follows.

$$\frac{d\theta_\mu}{dt} = \sum_{\nu(\neq\mu)} e^{-|r_{\nu\mu}|} \sin\left(\theta_{\nu\mu} + \alpha\,|r_{\nu\mu}| - c_1\right),$$

$$\frac{dr_\mu}{dt} = c_3 \sum_{\nu(\neq\mu)} e^{-|r_{\nu\mu}|} \frac{r_{\nu\mu}}{|r_{\nu\mu}|} \sin\left(\theta_{\nu\mu} + \alpha\,|r_{\nu\mu}| - c_2\right).$$

particle index:$\mu, \nu \in \{1, \cdots N\}$ with $\theta_{\nu\mu} := \theta_\nu - \theta_\mu$, $r_{\nu\mu} = r_\nu - r_\mu$ variables: $r_\mu \in Space, \theta \in [0, 2\pi] \approx S^1, t \in \mathbb{R}$ constants: $c_1, c_2 \in S^1, c_3 > 0, \alpha > 0$ We can choose any smooth manifold as a "Space". To make discussion simplifier and to compare the result of previous works, we chose 2 dimensional Euclidian space $\mathbb{R}^2$ for calculation and 2 dimensional torus T^2 for computer simulation. In this paper, we call a space as "plane" for this reason. When a correlation length is short enough in comparison to a system size, we can regard both case as an almost same situation. For our target phenomena, the aggregation wit target and spiral pattern wave, which is a local event, this treatment can be justified well.

2.2 Continuation of Swarm Oscillators

In this chapter, we propose a continuous version of Swarm Oscillators (a Vlasov equation of Swarm Oscillator). Spiral and Target pattern wave can be observed when adequately many particles are filled in the plane. Hence a continuation of the equation should be the reasonable first step of analysis. Stems from a large deviation theory, we can expect a same behavior as seen in the original discrete system. To be accurate, if we use this type of continuation,discrete one will converge weakly to the continuous one as $N \to \infty$. Here we introduce a density function ρ in $Space \times S^1$ as $\rho(t, r, \theta) = \sum_\mu \delta(r - r_\mu)\delta(\theta - \theta_\mu)$. With this density function, we can derive a following continuous version of Swarm Oscillators.

$$\frac{\partial \rho}{\partial t} = -D\zeta(t, r, \theta)e^{-i\theta} + c.c,$$

$$D := \frac{\tilde{c_3}e^{-i\tilde{c_2}}}{2}\frac{\partial}{\partial r}\rho(t, r, \theta)\frac{\partial}{\partial r} + \frac{e^{-i\tilde{c}_1}}{2}\frac{\partial}{\partial \theta}\rho(t, r, \theta)\frac{\partial}{\partial \theta},$$

$$\zeta(t, r) := \int dt' dr' d\theta' \rho(t, r', \theta')g(r' - r)e^{i\theta'} \qquad (1)$$

$g(r) := \exp((-1 + i\alpha)|r|)$. D operates linearly in a smooth function space, ζ is a mean field express the extracellular condition. And parameters are replaced as $\tilde{c}_1 = c_1$, $\tilde{c_2} = c_2 + \tan^{-1}(-\alpha) + \pi/2$, $\tilde{c_3} = c_3/\sqrt{1 + \alpha^2}$.

2.3 Multi Timescale Perturbation

Using a manipulation of multi timescale perturbation, we can divide a dynamical equation which describes the phase wave and the particle migration movement. By M.Iwasa and R.Ishiwata's work[4], the value of the parameter c_3 was estimated to be in the order of 10^{-1} in the case of *D.discoideum.* The parameter c_3 is a rate of a time scale difference between the cell movement and the phase change. Hence it's reasonable to consider that the dynamics of the phase is much faster than that of the cell migration.

According to the treatment of a multi timescale perturbation, we introduce a couple of different scale time variables $(t_0, t_1) = (t, c_3 t)$,which is fast clock and slow clock respectively. We consider that these two times t_0 and t_1 are independent, and we can expand the density function rho with c_3 as follows.

$$\rho(t_0, t_1, r, \theta) = \rho_0(t_0, t_1, r, \theta) + c_3\rho_0(t_0, t_1, r, \theta) + \cdots.$$

Comparing each order of the equation, we got following sequential equations. The 0th order is coincident with a locally coupled Kuramoto oscillators [5].

$$0th\ Order : \frac{\partial \rho_0}{\partial t_0} = -\frac{e^{-i\tilde{c}_1}}{2}\frac{\partial}{\partial \theta}\rho_0\frac{\partial}{\partial \theta}\tilde{g} \cdot \rho_0 + c.c,$$

$$1st\ Order : \frac{\partial \rho_1}{\partial t_0} + \frac{\partial \rho_0}{\partial t_1} = -\frac{e^{-i\tilde{c}_2}}{2}\frac{\partial}{\partial \theta}\rho_0\frac{\partial}{\partial \theta}\tilde{g} \cdot \rho_0 + c.c.$$

$\tilde{g}(r, \theta) := g(r)e^{-i\theta}$ and the product"$\cdot$" means a convolution integral with respect to r and θ. If we integrate the r.h.s of the 1st order equation with respect to the fast time t_0, it will diverse in a order of $O(t_0)$. As we can see in a numerical simulation, this type of divergence does not appear. To remove this unphysical divergence, we introduced a following renormalization group equation.[6]

$$RG\ equation : \frac{\partial \rho_0}{\partial t_1} = -\frac{e^{-i\tilde{c}_2}}{2}\frac{\partial}{\partial r}\rho_0\frac{\partial}{\partial r}g \cdot \rho_0 + c.c. \qquad (2)$$

This RG equation (2) decides a slow dynamics, that is, the transfer of the particles.

2.4 Solution of the 0th order eq.

Aforementioned, the 0th order equation is same with the Kuramoto equation. We derived following self consistent equation in the usual treatment of the Kuramoto equation. [5] We divided the density function into two states; a phase locked state and a phase drifting state, which are corresponding to the phase synchronized and the phase unsynchronized states respectively. Then we can derive a following closed equation, with that we can calculate the mean field $\zeta =: R\exp(i(\Theta + i\Omega))$. A parameter $\Omega(r)$ here we introduce is a macroscopic angular frequency. Center dot means convolution product.

$$R(r)e^{i\Theta(r)} = ie^{-i\tilde{c}_1}\int dr' S(r')g(r - r')e^{i\Theta(r')}$$
$$\left\{\frac{\Omega(r')}{R(r')} - \sqrt{\left(\frac{\Omega(r')}{R(r')}\right)^2 - 1}\right\}. \qquad (3)$$

S is a particle distribution in the plane at time t_1. To make an appearance simpler,we abbreviated the dependence of R and Θ on slow time variable t_1.

With a following ansatz of the wave form which was used by D.S.Cohen et.al[7], we got the solution of this equation (3).

$$spiral: R(r) = R(|r|), \Theta(r) = f(|r|) + \tan^{-1}\left(\frac{y}{x}\right),$$

$$target: R(r) = R(|r|), \Theta(r) = f(|r|).$$

For make our discussion and calculation easier, let's restrict our discussion to the case that particles are uniformly distributed in the plane. The solution of the equation (3) under this condition is shown in figure (1), figure (3) and (2). From these figures, we can find that $R(r)$ and $f(r) = \Theta(r) - \Omega t$ don't depend on $\tilde{c}_1$ and that $Omega$ can be approximated by one degree polynominal of $\tilde{c}_1$ in both spiral pattern and target pattern case.

We can deduct density dependence of $R(r)$, $f(r)$ and Ω from this result. Let (3) deformed to following form.

$$\tilde{R}(r)e^{i\Theta(r)} = \left(\frac{i\rho e^{-i\tilde{c}_1}}{\Omega}\right)\int dr' g(r-r')e^{i\Theta(r')} \left\{\frac{1}{\tilde{R}(r')} - \sqrt{\left(\frac{1}{\tilde{R}(r')}\right)^2 - 1}\right\}. \quad (4)$$

$\tilde{R}$ is defined by $\tilde{R}(r) = R(r)/\Omega$ Let the mean density rho substituted to $\chi\rho$, we can keep the equation same by replacing Ω with $\chi\rho$. And also the solution "$\tilde{R}$, Θ" keeps. This invariance means that R and Ω are proportional to the density ρ.

We note that an unti-direction spiral wave gives a same result because the equation (3) has a space inversion symmetry and the direction of the spiral is determined by the initial condition. On the other side, a solution of a negative phase velocity $\Omega < 0$ can also exist in the condition $\tilde{c}_1$ is large. Transformation $\tilde{c}_1 \to 2\pi - \tilde{c}_1$ changes a signature of a square root term of the equation 3. We can construct the solution of negative Ω solution by the inversion of $\tilde{c}_1$ from the solution of positive Ω solution. We also found the fact that phase velocity profile and wave length profile are almost same between the cases of spiral pattern wave and target pattern wave for some reason.

2.5 Particle Flux

Calculation of the particle flow is one of the most important information in order to compare the result with an experimental results. Here we propose how to calculate a vector field of a density weighted velocity, that is, a flux field. Substituting the phase dynamics factor which was calculated in the previous section to the RG equation2, we can calculate the mean field "ζ". The flux is determined by the gradient of ζ. See the equation (2). Substituting the result of ζ in previous sections to the equation

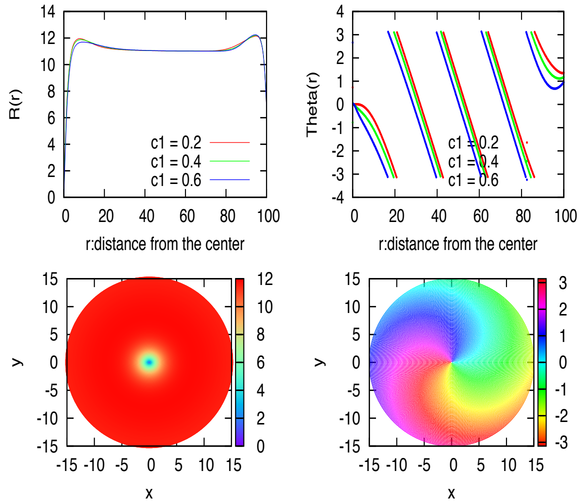

図 1: The above figure shows the c1 dependence of $R(|r|)$ and $f(|r|) = \Theta - \Omega t$ of the spiral wave. An above left figure is a graph of $R(|r|)$, an above right figure is $f(|r|) = \Theta(r,t) - \Omega t$, a below left figure a 2d graph of $R(r)$, and an below right figure is a 2d graph of $Theta(r) - \Omega t$. From these figures, we can see that the functions R and f are universal for any choice of the parameter $\tilde{c}_1$. And the center area of the $R(r)(|r|\ 0)$ is sharply narrow.

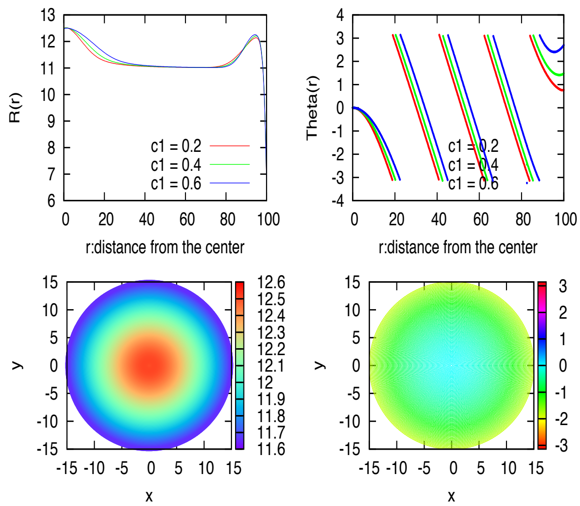

図 2: The above figure shows the c1 dependence of $R(|r|)$ and $f(|r|)$ of the target wave. From these figures, we can see an universality as is also seen in the spiral pattern wave. The center of the target pattern is ambiguous than that of the spiral pattern.

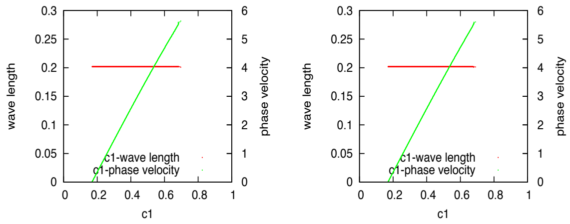

図 3: The above figure shows the c1 profile of wave length and phase velocity Ω. The left figure is those of spiral pattern wave and the right one is those of target pattern wave. From this figure, we can see that the phase velocity Ω can be fitted by one polynominal function of $\tilde{c}_1$ and that the wave length has a constant value for all $\tilde{c}_1$ in both wave case. It's notable that the wave length and phase velocity of both cases are almost same.

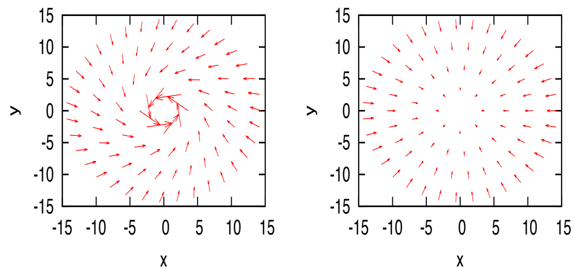

図 4: The figure shows the vector field of the particle velocity. The left figure is the flux vector field corresponding the spiral pattern wave, a right figure is that of the target pattern wave. From these figures, we can see, in case of spiral pattern wave, the flux is large near the center point of the wave, . On the contrary, in the case of target wave, that is small. And the flux is globally larger in the case of spiral pattern wave, than target one. This result implies that the particles can migrate more effectively with spiral wave pattern.

(2), we can get the flux. We show the result in figure (4). We can see the flux of the aggregation with spiral pattern wave is much faster than that of target pattern wave. This result imply that spiral wave pattern is more effective signal than target pattern wave for the aggregation process.

3 Discussion & Future Work

In this paper, we especially focus on the first term, so the study of the middle term is the most accesible future work. Through my calculation, the senario of the aggregation process especially the relation between the wave and migration, got clear.

We note that when we distribute the particles homogeneously in the plane, the first period of the aggregation process is driven by phase gradient.And in middle term of aggregation, when particles are assembling to the center of the pattern, the aggregation process is driven by the phase gradient in a far area from the center and by the gradient of the particle density near the center. In late term, the senario of aggregation is considered to be driven by only the density gradient. The particles are concentrated and emerged at the center. My continuous theory can be applied to the first and middle term of the aggregation process.

And aforementioned, we noted that my multi time scale perturbation theory is not completed. Replacing the constant Ω by some appropriate function $\Omega(t_1, r)$, In rough evaluation($S(r) \approx \rho + \nabla S(r_0) \cdot (r - r_0)$), Ω can be defined as satisfying the $\tilde{c}_1 - \Omega$ graph and proportional relationship between density and Ω. Here we propose a candidate of (R, Θ, Ω) for general value of the density ρ.

$$R(r;\tilde{c}_1)e^{i\Theta(r;\tilde{c}_1)} = \Omega(r;\tilde{c}_1)\tilde{R}_{flat}(r;\rho=1)e^{i\Theta_{flat}(r;\tilde{c}_1,\rho(r))}$$

$$\Omega(r;\tilde{c}_1) = \rho(r)\Omega_{flat}(\tilde{c}_1, \rho = 1),$$

$$\Theta(r;\tilde{c}_1) = \Theta_{flat}(\tilde{c}_1, \rho = 1),$$

With this hypothesis, we can get the equation in which phase dynamics are projected to the center attractor manifold. It's my future work to proof of this hypothesis or construct a solution of fast dynamics solution in another analytical way.

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