Cluster Formation in Swarm Oscillators Model

Masatomo Iwasa[1], Kazuki Iida [1,2], Dan Tanaka [1,3]

[1] Department of Complex Systems Science, Graduate School of Information Science, Nagoya University
[2] Department of Human and Artificial Intelligent Systems, Graduate School of Engineering, University of Fukui
[3] Precursory Research for Embryone Science and Technology (PRESTO), Japan Scinece and Technology Agency (JST)

Abstract

Swarm-oscillators model derived by one of the authors (DT) is a dynamical system for many interacting motile oscillators which exhibits various kinds of ordered structure. Here we particularly focus on cluster structures in the model. For one-dimensional system, we see all stable static structures in a particular but large set of parameters. For two-dimensional one, we relate parameter values in the model to the form of clusters.

---

1 Swarm Oscillators Model

Cluster formation is one of the main characteristics in self-organizing systems such as the traffic flow system, internal network of genes and proteins in a bacterium, simple robots employing swarm intelligence and nano-devices with bottom-up approach. Various models exhibiting clusters have been proposed and analyzed to date[1, 2, 3, 4, 5]. In this letter, we investigate one of them, Swarm oscillators model (simply referred to as SO model in what follows) derived by one of the authors (DT)[6], and discuss cluster structures arising in the model. SO model is a dynamical system which describes interacting motile oscillators as follows:

$$\dot{\psi}_i = \sum_{\{j|j\neq i\}} \mathrm{e}^{-|\vec{R}_{ji}|} \sin\left(\Psi_{ji} + \alpha|\boldsymbol{R}_{ji}| - c_1\right), \tag{1}$$

$$\dot{\boldsymbol{r}}_i = c_3 \sum_{\{j|j\neq i\}} \hat{\boldsymbol{R}}_{ji} \mathrm{e}^{-|\boldsymbol{R}_{ji}|} \sin\left(\Psi_{ji} + \alpha|\boldsymbol{R}_{ji}| - c_2\right). \tag{2}$$

Here, ψ_i and $\boldsymbol{r}_i$ denote the phase and the position of i-th oscillator respectively ($i = 1, \cdots, N$ for N oscillators system). The over-dot represents differentiation with respect to time. $\boldsymbol{R}_{ji} := \boldsymbol{r}_j - \boldsymbol{r}_i$, $\Psi_{ji} := \psi_j - \psi_i \pmod{2\pi}$, $\hat{\boldsymbol{R}}_{ji} := \boldsymbol{R}_{ji}/|\boldsymbol{R}_{ji}|$. There are four real parameters, $0 \leq c_1 < 2\pi$, $0 \leq c_2 < 2\pi$, $0 \leq c_3$ and $0 \leq \alpha$. A few results of numerical simulations are shown in Fig. 1 all of which show cluster structures although many other kinds of patterns emerge in this model.

2 Two-Oscillators System

To understand the collective behavior in SO model, the best strategy is to investigate the most basic version of SO model first of all, that is two-oscillators system. By setting $\delta := \psi_2 - \psi_1 \pmod{2\pi}$ and $\rho := r_2 - r_1$, we derive SO model for two-oscillators system as follows:

$$\dot{\delta} = -\mathrm{e}^{-\rho}\cos(\alpha\rho - c_1)\sin\delta, \tag{3}$$

$$\dot{\rho} = c_3\mathrm{e}^{-\rho}\sin(\alpha\rho - c_2)\cos\delta. \tag{4}$$

We immediately find the fixed points of this system, $(\delta_0,\ \rho_0)$, as follows:

$$(\delta_0,\ \rho_0) = \left(0,\ \frac{c_2 + p\pi}{\alpha}\right) \quad \text{or} \quad \left(\pi,\ \frac{c_2 + p\pi}{\alpha}\right),\ p \in \mathbb{Z}; \tag{5}$$

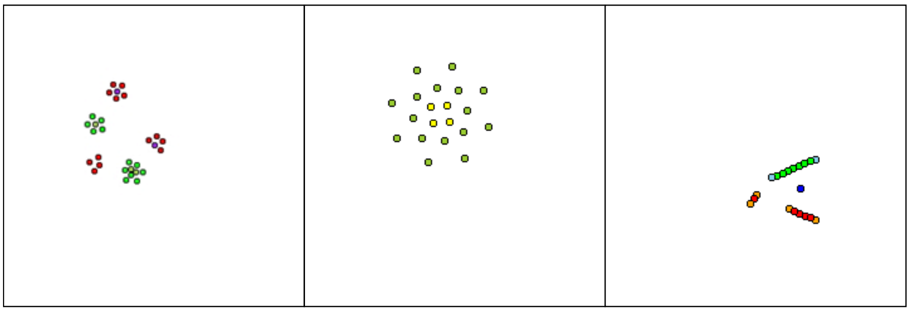

Figure1: Examples of two-dimensionally spreading cluster structure (left and middle) and an example of one-dimensionally spreading cluster structure (left) in SO model in two-dimensional space.

and

$$(\delta_0,\ \rho_0) = \left(\frac{\pi}{2},\ \frac{2c_1+\pi+2q\pi}{2\alpha}\right) \quad \text{or} \quad \left(\frac{3\pi}{2},\ \frac{2c_1+\pi+2q\pi}{2\alpha}\right),\ q \in \mathbb{Z}. \tag{6}$$

According to the linear stability analysis around each fixed point, we find out that stable fixed points appear if and only if $\cos(c_1-c_2)>0$. Those points are $(\delta_0,\ \rho_0)=(0,\ \frac{c_2+p\pi}{\alpha})$ for even p, and $(\delta_0,\ \rho_0)=(\pi,\ \frac{c_2+p\pi}{\alpha})$ for odd p. This means that the distance between oscillators takes discrete values labeled by p, and the phase difference takes 0 or π depending on the value of p. As we see later, this feature remains in many-oscillators system. In contrast, if $\cos(c_1-c_2)\leq 0$, both the phase difference and the distance continue to oscillate temporally.

3 One-Dimensional System

Next we concentrate on more complex but basic version, that is one-dimensional system. Particularly, we consider a simple case, $c_1 = c_2 =: c$. Results of numerical simulations are shown in Fig.2. We see fractal-like cluster structures there. By defining distances and phase differences between adjacent

Figure2: Examples of cluster patterns in one-dimensional swarm oscillators model.

oscillators, $\rho_i := r_{i+1}-r_i$ and $\delta_i := \psi_{i+1}-\psi_i$, SO model reads

$$\dot{\psi}_i = \sum_{j=1}^{i-1} \mathrm{e}^{-\sum_{k=j}^{i-1}\rho_k} \sin\left(\alpha\sum_{k=j}^{i-1}\rho_k - \sum_{k=j}^{i-1}\theta_k - c\right) + \sum_{j=i}^{N-1} \mathrm{e}^{-\sum_{k=i}^{j}\rho_k} \sin\left(\alpha\sum_{k=i}^{j}\rho_k + \sum_{k=i}^{j}\theta_k - c\right), \tag{7}$$

$$\dot{r}_i = -c_3\sum_{j=1}^{i-1} \mathrm{e}^{-\sum_{k=j}^{i-1}\rho_k} \sin\left(\alpha\sum_{k=j}^{i-1}\rho_k - \sum_{k=j}^{i-1}\theta_k - c\right) + c_3\sum_{j=i}^{N-1} \mathrm{e}^{-\sum_{k=i}^{j}\rho_k} \sin\left(\alpha\sum_{k=i}^{j}\rho_k + \sum_{k=i}^{j}\theta_k - c\right). \tag{8}$$

For the sake of clear discussion, we introduce the 'renormalized' expression of SO model. That is,

$$\begin{aligned} \dot{\psi}_i &= A_{i-1}^{-}\mathrm{e}^{-\rho_{i-1}}\sin(\alpha\rho_{i-1}-\theta_{i-1}-c-\kappa_{i-1}^{-}) + A_i^{+}\mathrm{e}^{-\rho_i}\sin(\alpha\rho_i+\theta_i-c-\kappa_i^{+}), &(9)\\ \dot{r}_i &= -c_3A_{i-1}^{-}\mathrm{e}^{-\rho_{i-1}}\sin(\alpha\rho_{i-1}-\theta_{i-1}-c-\kappa_{i-1}^{-}) + c_3A_i^{+}\mathrm{e}^{-\rho_i}\sin(\alpha\rho_i+\theta_i-c-\kappa_i^{+}), &(10)\\ & i=1,\cdots,N. \end{aligned}$$

Here, we define $A_0^- = A_N^+ = 0$ for clear and simple representation. Both A_i^- and κ_i^- $(i = 1, \cdots, N-1)$ are the functions with respect to $\{\rho_1, \ \ldots, \rho_{i-1}, \ \theta_1, \ \ldots, \ \theta_{i-1}\}$, and both A_i^+ and κ_i^+ $(i = 1, \cdots, N-1)$ are those with respect to $\{\rho_{i+1}, \ \ldots, \ \rho_{N-1}, \ \theta_{i+1}, \ \ldots, \ \theta_{N-1}\}$. Note that obviously $A_1^- = A_{N-1}^+ = 1$ and $\kappa_1^- = \kappa_{N-1}^+ = 0$. In other words, Eq. (9) and Eq. (10) indicate that interaction with oscillators further than nearest neighbors are renormalized into $\{A\}$ and $\{\kappa\}$. After a straightforward calculation, we find out that the fixed points can be formally written as

$$\rho_i = \frac{1}{2\alpha}\left[2c + n\pi + (\kappa_i^+ + \kappa_i^-)\right], \ n \in \mathbb{Z}; \tag{11}$$

$$\theta_i = \frac{1}{2}(\kappa_i^+ - \kappa_i^-) \ \text{ or } \ \frac{1}{2}(\kappa_i^+ - \kappa_i^-) + \pi. \tag{12}$$

Omitting the detailed discussion in this letter, since ρ_i is large enough in usual case, we can assume that $\{A\}$ and $\{\kappa\}$ is sufficiently close to 1 and 0 respectively. Under this condition, by ordinary linear stability analysis, it is shown that the phase difference between adjacent oscillators which takes distance labeled by even n or odd n is as nearly 0 or π respectively. In other words, the minimal clusters consist of oscillators whose distance labeled by $n = 0$ where phase of those oscillators are almost the same. The second smallest clusters consist of those minimal cluster all of which are placed apart from another in the distance labeled by $n = 1$. Then the phase difference between them is almost π. In the same way, we find that fractal-like cluster structures arise (Fig. 3).

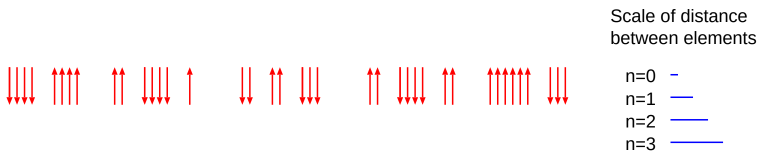

Figure3: Schematic figure of fractal-like cluster structure appearing in particular one-dimensional SO model.

4 Two-Dimensional System

What we are interested in next is cluster patterns in two-dimensional space. According to the numerical simulations, oscillators form two-dimensionally spreading cluster as shown on the left and the middle in Fig.1 for some parameter sets, or one-dimensional alignment as shown on the right in Fig. 1 for some other sets. To reveal the condition of the emergence of these patterns, we investigate three-oscillators SO model where regular triangle patterns and line pattens arise as shown in Fig. 4. Results of numerical

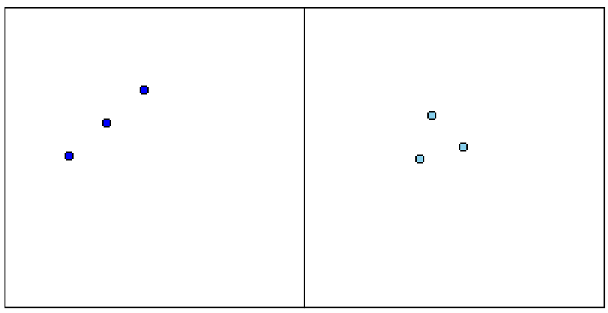

Figure4: The line alignment and the regular triangle arising in three-oscillators system.

simulations to see the dependence of the appearance of those patterns on c_1 are shown on the left in Fig. 5. This result reflects two-oscillators system, that is to say, static configurations arise only when $\cos(c_1 - c_2) > 0$, which corresponds to $c_1 < 2.57$ and $5.71 < c_1$ in Fig. 5. For the purpose stated above, we focus on the range $\cos(c_1 - c_2) > 0$. For three oscillators, structure formations are roughly explained as follows: In general, three distances between oscillators are different from each other. The closer oscillators interact stronger than the others. Then, the closer two pair go to stable fixed points as if they were two-oscillators system. In this way, an isosceles triangle are firstly formed. After that, oscillators move to a regular triangle or a line. A result of a numerical simulation supports this picture (right figure in Fig.5).

Next, we focus on the transition between regular triangle patterns and line patterns. By a straightforward calculation, we can immidiately show both line patterns and regular triangle patterns are solutions of SO model for all parameter values. Therefore, which pattern appears is determined by their stabilities. With ordinary linear stability analysis, we find out that regular triangle patterns are stable if and only if $\sin(c_1 - c_2 - \gamma) > 0$, where $\cos\gamma = \frac{1}{\sqrt{25c_3^2\alpha^2+1}}$ and $\sin\gamma = \frac{5c_3\alpha}{\sqrt{25c_3^2\alpha^2+1}}$. This condition corresponds to $c_1 < 2.73$ or $5.51 < c_1$ in Fig.5. In the same way, we can find out that line patterns are stable if and only if $\sin(c_1 - c_2) > 0$, which corresponds to $1.00 < c_1 < 4.14$ in Fig.5, under the appropriate approximation such that we neglect the interaction between both edges of the line.

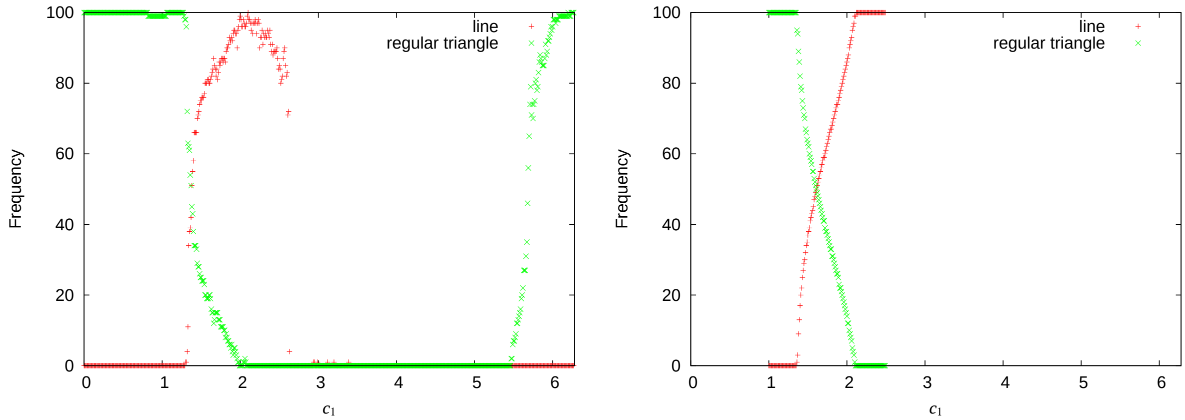

Figure5: (Left) The histogram of static line patterns and static regular triangle patterns for numerical simulations under the condition $c_2 = c_3 = \alpha = 1.00$ for 100 trial. Oscillators are randomly distributed initially but close to each other enough to construct minimal size of stable patterns. (Right) The same histogram for specific initial states in the range of $1.00 < c_1 < 2.50$ under the same set of parameters. Initial positions are isosceles triangle and all of initial phase are idintical. Its vertex angle is changed from $\frac{\pi}{3}$ to π in steps of $\frac{2\pi}{300}$.

Omitting the detailed explanation in this letter, we can observe by numerical simulation that oscillators form two-dimensionally spreading cluster structures (shown on the left and the middle in Fig. 1) in the range of regular triangle phase, or they form one-dimensionally spreading cluster structure (shown on the right in Fig. 1) in the range of line phase. Therefore, it is confirmed that the condition for the formations of regular triangle patterns and line patterns reflects to the cluster formation in more-oscillators system.

References

[1] T. Vicsek, A. Czirok, E. Ben-Jacob, I, Cohen and O. Shochet, Phys. Rev. Lett. **75**, 1226 (1995)

[2] U. Erdmann, W. Ebeling and V. S. Anishchenko, Phys. Rev. E **65**, 061106 (2002)

[3] T. Shibata and K. Kaneko, Physica D **181**, 197 (2003)

[4] A. S. Mihailov and D. H. Zanette, Phys. Rev. E **60** 4571 (1999)

[5] V. Gazi and K. M. Passino, IEEE Transactions on automatic control **48** 692 (2003)

[6] D. Tanaka, Phys. Rev. Lett. **99** 134103 (2007)